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On generators and defining relations of Yangians

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So far, two equivalent definitions of Yangians are known. The first one involves a finite set of generators and relations but some of the latter are too complicated, while the second one exploits infinite sequences of generators and relations of more convenient form. It is shown in this paper that only a finite part of these sequences suffices to define the structure of Yangians. In addition, we construct generators for the "Cartan part" of Yangian, which enjoy properties more similar to those of generators of a Cartan subalgebra of a simple Lie algebra.

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Drinfeld [3] introduced the notion of a quantum group, the most important examples being quantized enveloping algebras and Yangians. Both these types of quantum group are closely related to solutions of the quantum Yang–Baxter equation, hence thorough study of them is of interest.

So far, two equivalent definitions of Yangians are known. The first one, given by Drinfeld [3], involves a finite set of generators and relations but some of the latter are too complicated. Later, Drinfeld [4] (see also ref. [5]) suggested a more convenient set of generators and relations but this set is infinite. In the main theorem of the present paper we show that only a finite part of the last set suffices to define the structure of Yangians. In addition, we construct the generators of the "Cartan part" of Yangian, which enjoy properties similar to those of the generators of a Cartan subalgebra of a simple Lie algebra.

It should be noted that so far Yangians are introduced for simple Lie algebras only whereas it seems reasonable to introduce them in the case of affine Lie algebras as well. Still, it can be shown that one cannot expect simple infinite sequences of relations similar to those of ref. [4] to exist in the case of affine Lie algebras. Therefore it is important to be able to handle Yangians by using only a finite set of the relations introduced in ref. [4] (or their analogs). The results on "affine Yangians" will appear elsewhere.

For various results on the representation theory of Yangians and their relations with the theory of integrable quantum systems, see Drinfeld [3–5] and Chari and Pressley [1,2].

1. Main theorems

Let g be a finite-dimensional complex simple Lie algebra, let $g=n_+\oplus\eta\oplus n_-$ be a triangular decomposition of g, let Δ^+ be the set of positive roots and $\{\alpha_1, \ldots, \alpha_n\}$, $n=\operatorname{rank} g$, the corresponding set of simple roots, and let (a_{ij}) be the Cartan matrix of g.

Fix a non-zero invariant symmetric bilinear form (,) on g, and for each positive root α of g, choose root vectors $x_\alpha^\pm$ in the $\pm\alpha$ root spaces such that $(x_\alpha^+, x_\alpha^-)=1$. Then set $h_\alpha=[x_\alpha^+, x_\alpha^-]$, $h_i=h_{\alpha_i}$, $x_i^\pm=x_{\alpha_i}^\pm$.

Definition 1.1. Denote by $\bar{Y}(g)$ the algebra over $\mathbb{C}$ with generators $x_{i0}^\pm$, $x_{i1}^\pm$, $\tilde{h}_{i0}$, $\tilde{h}_{i1}$ $(1\leqslant i\leqslant n)$ and defining relations

$$[\tilde{h}_{i0}, \tilde{h}_{j1}]=0\,, \quad [\tilde{h}_{i0}, \tilde{h}_{j1}]=0\,, \quad [\tilde{h}_{i1}, \tilde{h}_{j1}]=0\,; \tag{1.1}$$

$$[\tilde{h}_{i0}, x_{j0}^\pm]=\pm(\alpha_i, \alpha_j)x_{j0}^\pm\,,$$

$$[\tilde{h}_{i1}, x_{j0}^\pm]=\pm(\alpha_i, \alpha_j)x_{j1}^\pm\,; \tag{1.2}$$

$$[x_{i0}^+, x_{j0}^-]=\delta_{ij}\tilde{h}_{i0}\,, \quad [x_{i1}^+, x_{i0}^-]=(\tilde{h}_{i1}+\tfrac{1}{2}\tilde{h}_{i0}^2)\,; \tag{1.3}$$

$$[x_{i1}^\pm, x_{j0}^\pm]=[x_{i0}^\pm, x_{j1}^\pm]\pm b_{ij}(x_{i0}^\pm x_{j0}^\pm+x_{j0}^\pm x_{i0}^\pm)\,, \tag{1.4}$$

where $b_{ij}=(\alpha_i, \alpha_j)/2$;

$$[x_{i0}^\pm, [x_{i0}^\pm, \ldots, [x_{i0}^\pm, x_{j0}^\pm]\cdots]]=0\,, \quad i\neq j\,, \tag{1.5}$$

$(1-a_{ij}$ commutators$)$;

$$[[\tilde{h}_{i1}, x_{i1}^+], x_{i1}^-]+[x_{i1}^+, [\tilde{h}_{i1}, x_{i1}^-]]=0\,. \tag{1.6}$$

Set for $k\in Z_+$,

$$x_{i,k+1}^\pm=\pm(\alpha_i, \alpha_i)^{-1}[\tilde{h}_{i1}, x_{ik}^\pm]\,, \quad h_{ik}=[x_{ik}^+, x_{i0}^-]\,.$$

Theorem 1.2. *The algebra $\bar{Y}(g)$ is isomorphic to the Yangian $Y(g)$, the isomorphism being defined by*

$$\bar{Y}(g)\ni\left\{\begin{matrix} x_{ik}^\pm \mapsto x_{ik}^\pm \\ h_{ik}\mapsto h_{ik}\end{matrix}\right\}\in Y(g)\,.$$

In other words, in $\bar{Y}(g)$ the following relations hold:

$$[h_{ik}, h_{jl}]=0\,; \tag{1.7}$$

$$[h_{i0}, x_{jl}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{jl}^{\pm} ; \tag{1.8}$$

$$[x_{ik}^{+}, x_{jl}^{-}] = \delta_{ij}h_{i,k+l} , \tag{1.9}$$

$$[h_{i,k+1}, x_{jl}^{\pm}] = [h_{ik}, x_{j,l+1}^{\pm}] \pm b_{ij}(h_{ik}x_{jl}^{\pm} + x_{jl}^{\pm} h_{ik}) ; \tag{1.10}$$

$$[x_{i,k+1}^{\pm}, x_{jl}^{\pm}] = [x_{ik}^{\pm}, x_{j,l+1}^{\pm}] \pm b_{ij}(x_{ik}^{\pm}x_{jl}^{\pm} + x_{jl}^{\pm}x_{ik}^{\pm}) ; \tag{1.11}$$

$$\sum_{\sigma} [x_{ik_{\sigma(1)}}^{\pm}, [x_{ik_{\sigma(2)}}^{\pm}, ..., [x_{ik_{\sigma(m)}}^{\pm}, x_{jl}^{\pm}]\cdots]] = 0 , \tag{1.12}$$

for $i \neq j$, where $m = 1 - a_{ij}$ and the sum is taken over all the permutations σ of $\{1, ..., m\}$.

Relations (1.7)–(1.12) *imply relations* (1.1)–(1.6).

Remark 1.3. Relations (1.7)–(1.12) are just the relations introduced by Drinfeld in ref. [4].

The proof of theorem 1.2 is based on the following lemma. To state it we introduce the unital algebra A with the generators h_j, x_j ($j \in \mathbb{Z}_+$) and the defining relations

$$[h_k, h_l] = 0 ; \tag{1.13}$$

$$[h_k, x_l] = [h_{k-1}, x_{l+1}] + \gamma(h_{k-1}x_l + x_l h_{k-1}) , \tag{1.14}$$

where $h_{-1} = 1$ and $\gamma \in \mathbb{R}$ is independent of k, l. Set

$$h(t) = \sum_{k \geqslant -1} h_k t^{-k-1} , \quad x(\tau) = \sum_{l \geqslant 0} x_l \tau^{-l-1} ,$$

and define $\tilde{h}_k \in A$ ($k \in \mathbb{Z}_+$) by the equality

$$\tilde{h}(t) := \sum_{k \geqslant 0} h_k t^{-k-1} = \ln h(t) . \tag{1.15}$$

Here the right-hand side is the expansion in powers of t in the vicinity of $t = +\infty$.

Lemma 1.4. *Let (1.13) hold for $j \leqslant p$, $l \leqslant p$, and let (1.14) hold for $k \leqslant p$ and $l \in \mathbb{Z}_+$. Then the following relation holds for $k \leqslant p$ and $l \in \mathbb{Z}_+$:*

$$[\tilde{h}_k, x_l] = 2\gamma x_{k+l} + 2 \sum_{\substack{0 \leqslant s \leqslant k-2 \\ k+s \text{ even}}} \frac{\gamma^{k+1-s}}{k+1} C_{k+1}^{s} x_{l+s} . \tag{1.16}$$

Proof. Due to (1.15), $\tilde{h}_k - h_k$ is a polynomial in h_0, h_1, ..., h_{k-1}. Hence, while deriving (1.16), we may and shall assume that (1.13), (1.14) hold for all k, $l \in \mathbb{Z}_+$. Now (1.14) can be rewritten as

$$h(t)x(\tau)=x(\tau)h(t)\,\frac{t-\tau+\gamma}{t-\tau-\gamma}\,;$$

therefore

$$[\tilde{h}(t),x(\tau)]=[\ln h(t),x(\tau)]=\ln\frac{t-\tau+\gamma}{t-\tau-\gamma}\,x(\tau)\,. \tag{1.17}$$

Simple calculations give

$$\begin{aligned}\ln\frac{t-\tau+\gamma}{t-\tau+\gamma}&=\ln(1-t^{-1}(\tau-\gamma))-\ln(1-t^{-1}(\tau+\gamma))\\&=-\sum_{k\geqslant 1}[t^{-k}(\tau-\gamma)^k-t^{-k}(\tau+\gamma)^k]/k\\&=\sum_{k\geqslant 1}\frac{t^{-k}}{k}\sum_{0\leqslant j\leqslant k}C_k^j\tau^{k-j}\gamma^j((-1)^{j+1}+1)\,;\end{aligned}$$

hence, (1.17) gives (1.16). □

Corollary 1.5. *Let (1.8) hold for* $l\in\mathbb{Z}_+$, $1\leqslant i,j\leqslant n$, *let (1.10) hold for* $k\leqslant p$, $l\in\mathbb{Z}_+$, *and* $1\leqslant i,j\leqslant n$, *let (1.7) hold for* $k,l\leqslant p$, $1\leqslant i=j\leqslant n$, *and define* $\tilde{h}_{ik}$ $(0\leqslant k\leqslant p,$ $1\leqslant i\leqslant n)$ *by*

$$\tilde{h}_i(t):=\sum_{k\geqslant 0}\tilde{h}_{ik}t^{-k-1}=\ln\left(1+\sum_{k\geqslant 0}h_{ik}t^{-k-1}\right). \tag{1.18}$$

Then for $k\leqslant p$, $l\in\mathbb{Z}_+$, $1\leqslant i,j\leqslant n$,

$$\begin{aligned}[\tilde{h}_{ik},x_{jl}^{\pm}]&=\pm(\alpha_i,\alpha_j)x_{j,l+k}^{\pm}\\&\quad\pm\sum_{\substack{0\leqslant s\leqslant k-2\\k+s\text{ even}}}2^{s-k}(\alpha_i,\alpha_j)^{k+1-s}\frac{C_{k+1}^s}{k+1}x_{j,l\pm s}^{\pm}\,.\end{aligned} \tag{1.19}$$

Remark 1.6. Theorem 1.2 being proved, the commutation relations (1.19) hold for all indices. Note that these relations, modulo terms of smaller second indices, are similar to the usual commutation relations in g:

$$[h_i,x_j^{\pm}]=\pm(\alpha_i,\alpha_j)x_j^{\pm}\,.$$

In addition, the $\tilde{h}_{jk}$ behave a bit nicer under the action of the coproduct in the Yangian $Y(g)$ (for the definition of the coproduct in $Y(g)$, see refs. [1–5]).

Proposition 1.7.

$$\Delta\tilde{h}_{ik}=\tilde{h}_{ik}\otimes 1+1\otimes\tilde{h}_{ik}\quad(\bmod\ Y^-H\otimes HY^+)\,, \tag{1.20}$$

where $Y^{\pm}(H)$ is a subalgebra of $Y(g)$, generated by $x^{\pm}_{ik}$ (by 1 and h_{jl}).

Proof. An easy induction shows (cf. ref. [1], proposition 1.6, and ref. [2], proposition 3.2) that

$$\Delta h_{ik} = h_{ik} \otimes 1 + 1 \otimes h_{ik} + \sum_{1 \leqslant j \leqslant k} h_{i,j-1} \otimes h_{i,k-j} \tag{1.21}$$

modulo $Y^- H \otimes HY^+$, and (1.20) follows easily from (1.18), (1.19) and (1.21). □

Remark 1.8. Formula (1.20) allows us to formulate the highest weight modules theory for Yangians similarity to the one for simple Lie algebras (cf. the highest weight modules theory for Yangians based on (1.21), in refs. [4,1,2]).

2. Proof of theorem 1.2

First we show that relations (1.7)–(1.12) imply relations (1.1)–(1.6); next we deduce from (1.1)–(1.6) some simple relations in (1.7)–(1.12); after that we prove by induction the relations (1.7)–(1.12) for $i=j$, and finally we prove (1.7)–(1.12) for $i \neq j$.

By comparing $(1.3)_2$ with (1.9), we see that (1.1)–(1.5) follow from (1.7)–(1.12) once we show that $(1.2)_2$ holds with

$$\tilde{h}_{i1} = h_{i1} - \tfrac{1}{2} h_{i0}^2 .$$

From (1.10) and (1.8), we deduce

$$\begin{aligned}
[\tilde{h}_{i1}, x^{\pm}_{j0}] &= [h_{i1}, x^{\pm}_{j0}] - \tfrac{1}{2} [h_{i0}^2, x^{\pm}_{j0}] \\
&= [h_{i0}, x^{\pm}_{j1}] \pm b_{ij}(h_{i0} x^{\pm}_{j0} + x^{\pm}_{j0} h_{i0}) \mp b_{ij}(h_{i0} x^{\pm}_{j0} + x^{\pm}_{j0} h_{i0}) \\
&= \pm (\alpha_i, \alpha_j) x^{\pm}_{j1} ,
\end{aligned}$$

and $(1.2)_2$ with $h_{i1} = h_{i1} - h_{i0}^2/2$ holds. Thus, (1.1)–(1.5) hold.

Formula (1.6) can be rewritten as

$$(\alpha_i, \alpha_i)^{-1} ([x^{+}_{i2}, x^{-}_{i0}] - [x^{+}_{i1}, x^{-}_{i1}]) = 0 ,$$

and hence follows from (1.7), (1.9).

Now we start deriving (1.7)–(1.12) from (1.1)–(1.6). From (1.1), (1.2) and the definition of $x^{\pm}_{jl}$, h_{ik}, it follows that

$$[h_{i0}, x^{\pm}_{jl}] = \pm (\alpha_i, \alpha_j) x^{\pm}_{jl} ; \tag{2.1}$$

$$[h_{i0}, h_{jl}] = 0 ; \tag{2.2}$$

$$[h_{i1}, x^{\pm}_{jl}] = \pm (\alpha_i, \alpha_j) x^{\pm}_{j,l+1} ; \tag{2.3}$$

$$h_{i1}=[x_{i1}^{+},x_{i0}^{-}]=[x_{i0}^{+},x_{i1}^{-}]\ . \tag{2.4}$$

By commutating both sides of (2.4) with $\tilde{h}_{i1}$ we obtain

$$h_{i2}=[x_{i2}^{+},x_{i0}^{-}]=[x_{i1}^{+},x_{i1}^{-}]=[x_{i0}^{+},x_{i2}^{-}]\ . \tag{2.5}$$

Now we see that (1.6) is equivalent to

$$[h_{i2},h_{i1}]=0\ . \tag{2.6}$$

As above we deduce from (1.2)

$$[h_{i1},x_{j0}^{\pm}]=[h_{i0},x_{j1}^{\pm}]\pm b_{ij}(h_{i0}x_{j0}^{\pm}+x_{j0}^{\pm}h_{i0})\ ,$$

and commutation with h_{j1}, due to $(1.1)_2$, gives

$$[h_{i1},x_{jl}^{\pm}]=[h_{i0},x_{j,l+1}^{\pm}]\pm b_{ij}(h_{i0}x_{jl}^{\pm}+x_{jl}^{\pm}h_{i0})\ . \tag{2.7}$$

To start the proof of (1.7), (1.9)–(1.11) for $i=j$, we note first that (1.7) with $k, l\geqslant 2$ follows from (2.2) and (2.6), formula (1.10) with $k=0$ is (2.7), and formula (1.11) with $k=l=0$ is just (1.4), which we can rewrite as follows:

$$[x_{i1}^{\pm},x_{i0}^{\pm}]=\pm\tfrac{1}{2}(\alpha_i,\alpha_i)(x_{i0}^{\pm})^2\ . \tag{2.8}$$

Lemma 2.1. *Let $s, p\in\mathbb{Z}_+$, let*

$$[h_{il},h_{ik}]=0\ ,\quad \text{for } k+l\leqslant s\ , \tag{2.9}$$

let $\tilde{h}_{i0}, \tilde{h}_{i1},\ldots,\tilde{h}_{is}$ be defined by (1.18), let (1.19) hold for $k\leqslant s$ and $l\in\mathbb{Z}_+$, and let

$$[x_{i,p+1}^{\pm},x_{ip}^{\pm}]=\pm\tfrac{1}{2}(\alpha_i,\alpha_i)(x_{ip}^{\pm})^2\ . \tag{2.10}$$

Then (1.11) *holds for $k=p+s$ and $l=p$:*

$$[x_{i,p+s+1}^{\pm},x_{ip}^{\pm}]=[x_{i,p+s}^{\pm},x_{i,p+1}^{\pm}]$$
$$\pm\tfrac{1}{2}(\alpha_i,\alpha_i)(x_{i,p+s}^{\pm}x_{ip}^{\pm}+x_{ip}^{\pm}x_{i,p+s}^{\pm})\ .$$

Proof. Note that (1.19) is of the form

$$[\tilde{h}_{ik},x_{il}^{\pm}]=\pm(\alpha_i,\alpha_i)x_{i,l+k}^{\pm}\pm\sum_{0\leqslant r\leqslant k-2}d_{kr}^{i}x_{i,l+r}^{\pm}\ , \tag{2.11}$$

set $\tilde{\tilde{h}}_{i0}=h_{i0}$, and define inductively

$$\tilde{\tilde{h}}_{ik}=\tilde{h}_{ik}-\sum_{0\leqslant r\leqslant k-2}d_{kr}^{i}\tilde{\tilde{h}}_{ir}\ ,\quad k=1,2,\ldots,s\ .$$

Then

$$[\tilde{\tilde{h}}_{ik},x_{il}^{\pm}]=\pm(\alpha_i,\alpha_i)x_{i,k+l}^{\pm}\ ,\quad k=0,1,\ldots,s\ ,\quad l\in\mathbb{Z}_+\ , \tag{2.12}$$

and

$$\tilde{h}_{ik} = h_{ik} + \text{polynomial in } h_{i0}, h_{i1}, \ldots, h_{i,k-1} ,$$
$$h_{ik} = \tilde{h}_{ik} + \text{(another) polynomial in } h_{i0}, h_{i1}, \ldots, h_{i,k-1} . \tag{2.13}$$

Using (2.12) and (2.10), we obtain

$$\begin{aligned}[x^{\pm}_{i,p+s+1}, x^{\pm}_{ip}] &= \pm(\alpha_i, \alpha_i)^{-1}[[\tilde{h}_{is}, x^{\pm}_{i,p+1}], x^{\pm}_{i}] \\ &= [x^{\pm}_{i,p+s}, x^{\pm}_{i,p+1}] \pm (\alpha_i, \alpha_i)^{-1}[\tilde{h}_{is}, [x^{\pm}_{i,p+1}, x^{\pm}_{ip}]] \\ &= [x^{\pm}_{i,p+s}, x^{\pm}_{i,p+1}] \pm (\alpha_i, \alpha_i)^{-1}[\tilde{h}_{is}, x^{\pm 2}_{ip}(\pm\tfrac{1}{2}(\alpha_i, \alpha_i))] \\ &= [x^{\pm}_{i,p+s}, x^{\pm}_{i,p+1}] \pm \tfrac{1}{2}(\alpha_i, \alpha_i)(x^{\pm}_{i,p+s}x^{\pm}_{ip} + x^{\pm}_{ip}x^{\pm}_{i,p+s}) .\end{aligned}$$

This proves the lemma. □

Due to (2.2), (2.7) and (2.8), the conditions of lemma 2.1 hold for $s \leqslant 1, p = 0$. Hence, the conclusion of lemma 2.1 gives

$$[x^{\pm}_{i2}, x^{\pm}_{i0}] = \pm\tfrac{1}{2}(\alpha_i, \alpha_i)(x^{\pm}_{i1}x^{\pm}_{i0} + x^{\pm}_{i0}x^{\pm}_{i1}) . \tag{2.14}$$

Take the + sign and commutate with x^{-}_{i0}:

$$\begin{aligned}[h_{i2}, x^{+}_{i0}] + [x^{+}_{i2}, h_{i0}] &= \tfrac{1}{2}(\alpha_i, \alpha_i)(h_{i1}x^{+}_{i0} + x^{+}_{i0}h_{i1} + x^{+}_{i1}h_{i0} + h_{i0}x^{+}_{i1}) \\ &= [h_{i1}, x^{+}_{i1}] - [h_{i0}, x^{+}_{i2}] + \tfrac{1}{2}(\alpha_i, \alpha_i)(h_{i1}x^{+}_{i0} + x^{+}_{i0}h_{i1}) .\end{aligned}$$

Hence,

$$[h_{i2}, x^{+}_{i0}] = [h_{i1}, x^{+}_{i1}] + \tfrac{1}{2}(\alpha_i, \alpha_i)(h_{i1}x^{+}_{i0} + x^{+}_{i0}h_{i1}) , \tag{2.15}$$

and analogously,

$$[h_{i2}, x^{-}_{i0}] = [h_{i1}, x^{-}_{i1}] - \tfrac{1}{2}(\alpha_i, \alpha_i)(h_{i1}x^{-}_{i0} + x^{-}_{i0}h_{i1}) . \tag{2.16}$$

Since $[\tilde{h}_{i1}, h_{i2}] = 0$, we deduce from (2.15), (2.16):

$$[h_{i2}, x^{\pm}_{il}] = [h_{i1}, x^{\pm}_{i,l+1}] \pm \tfrac{1}{2}(\alpha_i, \alpha_i)(h_{i1}x^{\pm}_{il} + x^{\pm}_{il}h_{i1}) ; \tag{2.17}$$

in addition, we see that (2.11)–(2.13) hold for $k \leqslant 2$.

Lemma 2.2. *Formula (1.11) holds for* $k, l \in \mathbb{Z}_+, j \leqslant i$.

Proof. Set

$$\begin{aligned}X^{\pm}(i, k, l) = [x^{\pm}_{i,k+1}, x^{\pm}_{il}] - [x^{\pm}_{ik}, x^{\pm}_{i,l+1}] \\ \mp\tfrac{1}{2}(\alpha_i, \alpha_i)(x^{\pm}_{ik}x^{\pm}_{il} + x^{\pm}_{il}x^{\pm}_{ik}) ,\end{aligned}$$

and note that (2.12) with $k = 1, 2$ gives the following implications:

$$X^{\pm}(i,p,l)=0$$
$$\Rightarrow X^{\pm}(i,p+1,l)+X^{\pm}(i,p,l+1)=0\,,$$
$$X^{\pm}(i,p+2,l)+2X^{\pm}(i,p+1,l+1)+X^{\pm}(i,p,l+2)=0\,; \quad (2.18)$$
$$X^{\pm}(i,p,l)=0 \quad\Rightarrow\quad X^{\pm}(i,p+2,l)+X^{\pm}(i,p,l+2)=0\,. \quad (2.19)$$

Hence,

$$X^{\pm}(i,p,l)=0 \quad\Rightarrow\quad X^{\pm}(i,p+1,l+1)=0\,. \quad (2.20)$$

Now (2.8), (2.14) and (2.20) give

$$X^{\pm}(i,k,l)=0 \quad \text{for } 0\leqslant l<k\leqslant 2\,. \quad (2.21)$$

Suppose, (2.21) holds for $0\leqslant l<k\leqslant s$, where $s\geqslant 2$. Applying (2.18) with $p=s'$ and (2.19) with $p=s'-1$, $0\leqslant s'\leqslant s$, we obtain (2.21) for $0<l<k\leqslant s+1$; if we now apply (2.18) with $p=s$ and $l=0$, we obtain (2.21) for $0\leqslant l<k\leqslant s+1$. Thus, the lemma is proved. □

Now we shall prove the set of formulae

$$[h_{ik},h_{il}]=0\,, \quad k+l\leqslant s,\ k\geqslant 1\,; \quad (2.22)$$
$$[x_{is}^{+},x_{i0}^{-}]=[x_{i,s-1}^{+},x_{i1}^{-}]=\cdots=[x_{i0}^{+},x_{is}^{-}]\,; \quad (2.23)$$
$$[h_{ik},x_{il}^{\pm}]=[h_{i,k-1},x_{i,l+1}^{\pm}]$$
$$\pm\tfrac{1}{2}(\alpha_i,\alpha_i)(h_{i,k-1}x_{il}^{\pm}+x_{il}^{\pm}h_{i,k-1})\,, \quad k\geqslant 1,\ l\geqslant 0,\ k+l\leqslant s\,, \quad (2.24)^{\pm}$$

by using induction in s. For $s=3$, these formulae have already been proved. Suppose they hold for $s=r$, and consider the case of odd $r=2p-1$ first.

Since $[h_{ip},h_{il}]=0$ for $l\leqslant p$, we deduce from $(2.24)^{\pm}$:

$$[h_{ip},x_{im}^{\pm}]=[h_{i,p-1},x_{i,m+1}^{\pm}]\pm\tfrac{1}{2}(\alpha_i,\alpha_i)(h_{i,p-1}x_{im}^{\pm}+x_{im}^{\pm}h_{i,p-1})$$

for $m\in\mathbb{Z}_+$; hence, (2.11)–(2.13) hold with $k=p$ and $l\in\mathbb{Z}_+$, and

$$0=[h_{ip},h_{ip}]=[h_{ip},\tilde{\tilde{h}}_{ip}]=[[x_{ip}^{+},x_{i0}^{-}],\tilde{\tilde{h}}_{ip}]$$
$$=-(\alpha_i,\alpha_i)\{[x_{i,2p}^{+},x_{i0}^{-}]-[x_{i,p}^{+},x_{i,p}^{-}]\}\,. \quad (2.25)$$

Further, by commutating $\tilde{h}_{i1}$ with both sides of (2.23) with $s=2p-1$, we obtain

$$[x_{i,2p}^{+},x_{i0}^{-}]-[x_{i,2p-1}^{+},x_{i1}^{-}]$$
$$=[x_{i,2p-1}^{+},x_{i1}^{-}]-[x_{i,2p-2}^{+},x_{i,2}^{-}]=\cdots$$
$$=[x_{i1}^{+},x_{i,2p-1}^{-}]-[x_{i0}^{+},x_{i,2p}^{-}]\,. \quad (2.26)$$

By comparing (2.25) and (2.26), we conclude that (2.23) holds for $s=r+1=2p$ and that for $q<p$ the following equality holds:

$$[h_{i,r-q}, h_{i,q+1}] = [[x^+_{i,r-q}, x^-_{i0}], \tilde{\tilde{h}}_{i,q+1}]$$
$$= [x^+_{i,r+1}, x^-_{i0}] - [x^+_{i,r-q}, x^-_{i,q+1}] = 0 .$$

Hence, (2.22) holds for $s=r+1$. Finally, by commutating $\tilde{h}_{i1}$ with $(2.24)^\pm$ with $s=r$ and taking into account that (2.22) holds for $s \leqslant r+1$, we obtain $(2.24)^\pm$ with $s=r+1$. Thus, we have made an inductive step for r odd.

Now, let $r=2p$ be even. By commutating $\tilde{h}_{i1}$ with (2.23) with $s=2p$, we obtain

$$\begin{aligned}&[x^+_{i,2p+1}, x^-_{i0}] - [x^+_{i,2p}, x^-_{i1}]\\ &= [x^+_{i,2p}, x^-_{i1}] - [x^+_{i,2p-1}, x^-_{i2}] = \cdots\\ &= [x^+_{i1}, x^-_{i,2p}] - [x^+_{i0}, x^-_{i,2p+1}] .\end{aligned} \tag{2.27}$$

Due to (2.23), $[h_{ip}, h_{il}]=0$ for $l \leqslant p$; therefore we can define $\tilde{\tilde{h}}_{ip}$ [see (2.11)–(2.13)] and obtain

$$\begin{aligned}[h_{i,p+1}, h_{ip}] &= [h_{i,p+1}, h_{ip}] = [[x^+_{i,p+1-q}, x^-_{iq}], h_{ip}]\\ &= -(\alpha_i, \alpha_i)([x^+_{i,2p+1-q}, x^-_{iq}] - [x^+_{i,p+1-q}, x^-_{i,p+q}]) .\end{aligned} \tag{2.28}$$

On the other hand, using $(2.24)^\pm$ several times with $k+l \leqslant 2p$ gives

$$\begin{aligned}[h_{i,p+1}, h_{ip}] &= [[h_{i,p+1}, x^+_{i,p-1}], x^-_{i1}] + [x^+_{i,p-1}, [h_{i,p+1}, x^-_{i1}]]\\ &= [[h_{i0}, x^+_{i,2p}], x^-_{i1}]\\ &\quad + \Big[\sum_{0\leqslant j\leqslant p} (h_{i,p-j}x^+_{i,p+j-1} + x^+_{i,p+j-1}h_{i,p-j}), x^-_{i1}\Big]\\ &\quad + [x^+_{i,p-1}, [h_{i0}, x^-_{i,p+2}]]\\ &\quad + \Big[x^+_{i,p-1}, \sum_{0\leqslant j\leqslant p} (h_{i,p-j}x^-_{i,j+1} + x^-_{i,j+1}h_{i,p-j})\Big]\\ &= (\alpha_i, \alpha_i)([x^+_{i,2p}, x^-_{i1}] - [x^+_{i,p-1}, x^-_{i,p+2}])\\ &\quad + \sum_{0\leqslant j\leqslant p} \{[h_{i,p-j}x^+_{i,p+j-1} + x^+_{i,p+j-1}h_{i,p-j}, x^-_{i1}]\\ &\quad + [x^+_{i,p-1}, h_{i,p-j}x^-_{i,j+1} + x^-_{i,j+1}h_{i,p-j}]\} .\end{aligned} \tag{2.29}$$

In $Y(g)$, the left-hand side in (2.29) is equal to zero and so is the first term on the right-hand side; therefore the sum over j is zero as well. But this sum can be represented as a sum of monomials with the property that the sum of the second indices is not greater than $2p$; since by the induction hypothesis the sets of such vanishing sums in $Y(g)$ and $\bar{Y}(g)$ coincide, we see that in $\bar{Y}(g)$, (2.29) may be rewritten as

$$[h_{i,p+1}, h_{ip}] = (\alpha_i, \alpha_i)([x^+_{i,2p}, x^-_{i1}] - [x^+_{i,p-1}, x^-_{i,p+2}]) .$$

By comparing this equality with (2.28), we find $[h_{i,p+1}, h_{ip}]=0$, and (2.23) holds for $s=2p+1=r+1$. Now, the proof of (2.22) and (2.24) for $s=r+1$ is concluded just as in the case r odd above.

Thus, we have proved all the relations (1.7)–(1.11) for $j=i$; below, we prove (1.7)–(1.12) for $i\neq j$.

First we prove (1.11). For $k=l=0$ it is just (1.4). If $(\alpha_i, \alpha_j)=0$, then by commutating successively with $\tilde{h}_{i1}$ we obtain (1.11) for all $k\geqslant 0$ and $l=0$; then, by commutating with $\tilde{h}_{j1}$, we obtain (1.11) for all $k, l\in\mathbb{Z}_+$.

If $(\alpha_i, \alpha_j)\neq 0$, we commutate

$$X^\pm(i,j;k,l) := [x^\pm_{i,k+1}, x^\pm_{jl}] - [x^\pm_{ik}, x^\pm_{j,l+1}] \mp b_{ij}(x^\pm_{ik}x^\pm_{jl} + x^\pm_{jl}x^\pm_{ik})$$

with $\tilde{h}_{i1}$ and independently with $\tilde{h}_{j1}$ to obtain: if $X^\pm(i,j;k,l)=0$, then $(X^\pm(i,j;k+1,l), X^\pm(i,j;k,l+1))$ is a solution to the homogeneous system of two equations with the determinant

$$\begin{vmatrix} (\alpha_i, \alpha_i) & -(\alpha_i, \alpha_j) \\ (\alpha_j, \alpha_i) & -(\alpha_j, \alpha_j) \end{vmatrix} \neq 0 . \tag{2.30}$$

Hence,

$$X^\pm(i,j;k,l)=0 \quad\Rightarrow\quad X^\pm(i,j;k+1,l)=0 , \quad X^\pm(i,j;k,l+1)=0 .$$

Since $X^\pm(i,j;0,0)=0$, we deduce (1.11) for all k, l.

Relation (1.9) is proved similarly: if $(\alpha_i, \alpha_j)=0$ then we commutate $[x^+_{i0}, x^-_{j0}]=0$ with $\tilde{h}_{i1}$ and obtain $[x^+_{ik}, x^-_{j0}]=0$ for $k\in\mathbb{Z}_+$; after that, we commutate with $\tilde{h}_{j1}$ and obtain (1.9) for all $k, l\in\mathbb{Z}_+$. If $(\alpha_i, \alpha_j)\neq 0$, we commutate $[x^+_{ik}, x^-_{jl}]=0$ with $\tilde{h}_{i1}$ and independently with $\tilde{h}_{j1}$ to obtain

$$[x^+_{i,k+1}, x^-_{jl}]=0 , \quad [x^+_{i,k}, x^-_{j,l+1}]=0 .$$

Hence, the equality $[x^+_{i0}, x^-_{j0}]=0$ $(i\neq j)$ yields (1.9) for all k, l.

Now, we prove (1.10) for the + sign; the proof for the − sign is similar:

$$\begin{aligned}[h_{i,k+1}, x^+_{jl}] &= [[x^+_{i,k+1}, x^-_{i0}], x^+_{jl}] = [[x^+_{i,k+1}, x^+_{jl}], x^-_{i0}] \\ &= [[x^+_{ik}, x^+_{j,l+1}], x^-_{i0}] + b_{ij}[x^+_{ik}x^+_{jl} + x^+_{jl}x^+_{ik}, x^-_{i0}] \\ &= [h_{ik}, x^+_{j,l+1}] + b_{ij}(h_{ik}x^+_{jl} + x^+_{jl}h_{ik}) .\end{aligned}$$

To prove (1.7) we first note that the $\tilde{h}_{ik}$ satisfy (1.19) not only for $i\neq j$ (as was shown before) but for $i\neq j$ as well. Indeed, if we substitute in the commutation relations involving (i,j), $((\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))^{k+1}h_{ik}$ for h_{ik} and $((\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))^l x^\pm_{jl}$ for $x^\pm_{jl}$, we will get the commutation relations involving $i=j$. But this substitution, due to (1.17), replaces h_{ik} with $((\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))^{k+1}h_{ik}$; hence, (1.19) with $i=j$ gives (1.19) with $i\neq j$.

By using (1.18), (1.19), we can construct $\tilde{h}_{ij,k}$ ($k=0, 1, ...$) such that

$$[\tilde{h}_{ij,k}, x^{\pm}_{jl}] = \pm(\alpha_i, \alpha_j) x^{\pm}_{j,k+l}, \tag{2.31}$$

$$\tilde{h}_{ij,k} = h_{ik} + \text{polynomial in } h_{i0}, ..., h_{i,k-1} \tag{2.32}$$

(cf. proof of lemma 2.2). Using $(1.2)_1$ and (2.1), we get

$$[h_{i0}, h_{jl}] = [h_{i0}, [x^{+}_{jl}, x^{-}_{j0}]] = (\alpha_i, \alpha_j)^{-1}(h_{jl} - h_{jl}) = 0\,;$$

therefore (1.7) is proved for $k=0$ and all $l\in\mathbb{Z}_+$. Suppose (1.7) holds for $k=r$. We want to show that it holds for $k=r+1$. Due to (2.31) and the induction hypothesis we may substitute $\tilde{h}_{ij,k}$ for h_{ik}; after that (2.32) allows us to prove (1.7) like we did in the case $k=0$.

Now all that is left to do is to prove (1.12). If $(\alpha_i, \alpha_j)=0$, then (1.5) is $[x^{\pm}_{i0}, x^{\pm}_{j0}]=0$, and by commutating successively with $\tilde{h}_{i1}$ and $\tilde{h}_{j1}$, we obtain $[x^{\pm}_{ik}, x^{\pm}_{jl}]=0$, for all $k, l\in\mathbb{Z}_+$. This means that (1.12) holds.

Now, let $(\alpha_i, \alpha_j)\neq 0$. For $l\in\mathbb{Z}_+$ and a non-increasing tuple $k=(k_1, k_2, ..., k_m)$, where $k_j\in\mathbb{Z}_+$ and $m=1-a_{ij}$, denote by $X^{\pm}(k; l)$ the left-hand side of relation (1.12).

First we show that $X^{\pm}(0; l)=0$ for all $l\in\mathbb{Z}_+$, i.e., (1.12) holds for $k=(0, ..., 0)$ and all $l\in\mathbb{Z}_+$. For $l=0$, it is just (1.4). Assume $X^{\pm}(0; l)=0$ for $l\leqslant r$ and commutate with $\tilde{h}_{i1}$ and with $\tilde{h}_{j1}$. We get a system of two homogeneous linear equations with the unknowns $X^{\pm}((1, 0, ..., 0); r)$, $X^{\pm}(0; r+1)$ and the determinant (2.30). Hence, $X^{\pm}(0; l)=0$ for $l\in\mathbb{Z}_+$.

Finally, we use induction on the number s of the first zero entry k_i of k. For $s=0$, we have $X^{\pm}(k; l)=X^{\pm}(0; l)=0$. Assume $X^{\pm}(k; l)=0$ for all k with $s=s(k)\leqslant r$ and all $l\in\mathbb{Z}_+$. By commutating with h_{ip} and using the induction hypothesis, we get $X^{\pm}(k'; l)=0$ for all l and $k'=k'(k, p)$ which are obtained from $(k_1, ..., k_r, p, 0, ..., 0)$ by reordering. Hence, $X^{\pm}(k; l)=0$ for all k with $s=s(k)\leqslant r+1$ and all $l\in\mathbb{Z}_+$, and (1.12) is proved. □

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References

[1] V. Chari and A.N. Pressley, Yangians and R-matrices, L'Enseignement Mathématique 36 (1990) 267–302.

[2] V. Chari and A.N. Pressley, Fundamental representations of Yangians, Tata preprint (1990).

[3] V.G. Drinfeld, Hopf algebras and the quantum Yang–Baxter equation, Dokl. Akad. Nauk SSSR 283 (1985) 254–258 [Sov. Math. Dokl. 32 (1985)].

[4] V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, Dokl. Akad. Nauk SSSR 296 (1987) 212–216 [Sov. Math. Dokl. 36 (1988)].

[5] V.G. Drinfeld, Quantum groups, in: Proc. Intern. Congr. Math. (Berkeley, 1988), Vol. 1, pp. 798–820.